Math 246B Lecture 27 Notes

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1 Radial Limits of Harmonic Functions on the Disc

1.1 Radial limits of harmonic functions on the disc

Let $\mathcal{P} : \mathcal{M}(\partial D) \to h^1$, the set of all harmonic functions u in D such that $\int_{|z|=1} |u(rz)|\,|dz| \leq C$ for all r, send $\mu \mapsto \mathcal{P}\mu = u$. We showed last time that this is a homeomorphism.

Theorem 1.1. *Let $u \in h^1$, and consider the Lebesgue decomposition of the representing measure μ: $d\mu = f/(2\pi)\,|dz| + d\lambda$, where $f \in L^1(\partial D)$, and $d\lambda$ is singular with respect to $|dz|$.*

1. *Then for a.e. $z \in \partial D$, the radial limit $\lim_{r \to 1} u(rz)$ exists and equals $f(z)$.*
2. *If $d\mu = f/(2\pi)|dz|$, is absolutely continuous and $u(z) = \int_{|w|=1} P(z,w)\,d\mu(w)$, then $u_r \to f$ in $L^1(\partial D)$.*

Proof. Write

$$u(z) = \int_{|w|=1} P(z,w)\,d\mu(w) = \int_{[-\pi,\pi)} P(z, r^{i\varphi})\,d\mu(\varphi).$$

Recall that for a.e. $\varphi \in \mathbb{R}$, we have by the Lebesgue differentiation theorem that

$$\frac{1}{\rho} \int_{\varphi-\rho}^{\varphi+\rho} |f(e^{it}) - f(e^{i\varphi})|\,dt \xrightarrow{\rho \to 0} 0,$$

$$\frac{1}{\rho} \int_{[\varphi-\rho,\varphi+\rho]} |d\lambda(t)| \to 0.$$

We claim that if $\varphi \in \mathbb{R}$ is as above, then $\lim_{r \to 1} u(re^{i\varphi})$ exists and equals $f(e^{i\varphi})$. We may assume that $\varphi = 0$ and $f(1) = 0$. Then

$$\frac{1}{\rho} \int_{-\rho}^{\rho} |f(e^{it})|\,dt \to 0, \qquad \frac{1}{\rho} \int_{[-\rho,\rho]} |d\lambda(t)| \to 0.$$

It suffices to show that if $|nu$ is a measure such that $(1/\rho)\int_{[-\rho,\rho]}|d\nu(t)|\to 0$ as $\rho\to 0$, then

$$\int P(x,e^{it})\,d\nu(t)\xrightarrow{x\to 1^-}0,\qquad x\in\mathbb{R}.$$

Here,

$$\int_{\pi/2\le|t|\le\pi}P(x,e^{it})\,d\nu(t)$$

since $P(x,e^{it})\to 0$ uniformly. Write $\delta=1-x$, and consider

$$\int_{|t|\le\ pi/2}P(x,e^{it})\,d\nu(t)=\int_{\sqrt{c\delta}\le|t|\le\pi/2}P(x,e^{it})\,d\nu(t)+\int_{|t|\le\sqrt{c\delta}}P(x,e^{it})\,d\nu(t).$$

Here, $C>0$ is a large constant to be chosen later. When $\sqrt{C\delta}\le|t|\le|\pi/2|$,

$$P(x,e^{it})=\frac{1-x^2}{|x-e^{it}|^2}=\frac{2\delta-\delta^2}{(x-\cos(t))2+\sin^2(t)}\le\frac{2\delta}{\sin^2(t)}\le\frac{\pi^2\delta}{t^2}\le\frac{\pi^2\delta}{C\delta}=\frac{\pi^2}{C}.$$

Given $\varepsilon>0$, we get (taking C large but fixed)

$$\left|\int_{\sqrt{C\delta}\le|t|\le\pi/2}P(x,e^{it})\,d\nu(t)\right|\le\varepsilon$$

for all small $\delta>0$.

Let $\delta_1=\sqrt{C\delta}$, and let

$$\varphi(t)=P(x,e^{it})=\frac{1-x^2}{1+x^2-2x\cos(t)}.$$

Then $\varphi>0$, φ is even, and φ is decreasing on $[0,\pi]$. It remains to understand

$$\int_{|t|\le\sqrt{C\delta}}P(x,e^{it})\,d\nu(t)=\int_{|t|\le\delta_1}\varphi(t)\,d\nu(t).$$

We have

$$\int_{[-\rho,\rho]}|d\nu(t)|\le\varepsilon\rho,\qquad 0<\rho\le\delta_1.$$

Write

$$\varphi(t)=\varphi(\delta_1)+\int_{\delta_1}^{t}\varphi'(s)\,ds=\varphi(\delta_1)-\int_0^{\delta_1}H(s-t)\varphi'(s)\,ds,$$

where

$$H(\tau)=\begin{cases}1 & \tau>0\\ 0 & \tau\le 0\end{cases}$$

is the Heaviside function. Consider

$$\int_{[0,\delta_1]} \varphi(t)\, d\nu(t) = \varphi(\delta_1) \underbrace{\int_{[0,\delta-1]} d\nu(t)}_{\leq \varepsilon\delta_1} - \int_{[0,\delta_1]} \left(\int_0^{\delta_1} H(s-t)\varphi'(s)\, ds \right) d\nu(t).$$

Then

$$\begin{aligned} \left| \int_{[0,\delta_1]} \varphi(t)\, d\nu(t) \right| &\leq \varphi(\delta_1)\varepsilon\delta_1 - \int_0^{\delta_1} \varphi'(s) \left(\int_{[0,\delta_1]} H(s-t)\, |d\nu(t)| \right) ds \\ &\leq \varphi(\delta_1)\varepsilon\delta_1 - \int_0^{\delta_1} \varphi'(s) \underbrace{\left(\int_{[0,s]} |d\nu(t)| \right)}_{\leq \varepsilon s} ds \end{aligned}$$

Integrate by parts.

$$\begin{aligned} &\leq \varphi(\delta_1)\varepsilon\delta - 1 - \varepsilon \left[\varphi(s)s\right]_0^{\delta_1} + \varepsilon \int_0^{\delta_1} \varphi(s)\, ds \\ &= \varepsilon \int_0^{\delta_1} \varphi(s)\, ds \\ &\leq \varepsilon. \end{aligned}$$

The contribution of $[-\delta, 0]$ is estimated similarly. We get

$$u(x) = \int P(x, e^{it})\, d\nu(t) \xrightarrow{x \to 1^-} 0.$$

For the 2nd part of the theorem, given $\varepsilon >$, let $\psi \in C(\partial D)$ be such that $\|f - \psi\|_{L^1} \leq \varepsilon$. If we write $u = \mathcal{P}f$, then

$$\|(\mathcal{P}f)_r - f\|_{L^1} \leq \underbrace{\|(\mathcal{P}f)_r - (\mathcal{P}\psi)_r\|_{L^1}}_{\leq \|\mathcal{P}(f-\psi)\|_{h^1} \leq \|f-\psi\|_{L^1} \leq \varepsilon} + \underbrace{\|(\mathcal{P}\psi)_r - \psi\|_{L^1}}_{\to 0 \text{ uniformly on } \partial D} + \varepsilon.$$

We get $u_r = (\mathcal{P}f)_r \to f$ in L^1. $\square$

1.2 The Riesz-Riesz theorem

Let $H^1 = \mathrm{Hol}(D) \cap h^1$ (the **Hardy space**). It can be show that the representing measure of and H^1 function is absolutely continuous.

Theorem 1.2 (F. and M. Riesz[1]). *Let μ be a measure on ∂D such that $\int_{[0,2\pi)} e^{int}\, d\mu(t) = 0$ for $n = 1, 2, \ldots$ (i.e. the negative Fourier coefficients of μ vansish). Then μ is absolutely continuous.*

[1] These two were brothers. This is the only collaboration between them.

Proof. Here is the idea. Let $f = \mathcal{P}\mu \in h^1$. The vanishing of the Fourier coefficients implies that $f \in \mathrm{Hol}(D)$. So μ is absolutely continuous. $\square$